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On classical orthogonal polynomials and differential operators

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Abstract
It is well known that the four families of classical orthogonal polynomials (Jacobi, Bessel, Hermite and Laguerre) each satisfy an equation $\mathcal{F}P_n(x) = \lambda_n P_n(x), n \geqslant 0$, for an appropriate second-order differential operator $\mathcal{F}$. In this paper it is shown that any linear differential operator $\mathcal{U}$ which has the Jacobi, Bessel, Hermite or Laguerre polynomials as eigenfunctions has to be a polynomial with constant coefficients in the classical second-order operator $\mathcal{F}$.

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1. Introduction

The systems of orthogonal polynomials associated with the names of Hermite, Laguerre, Bessel and Jacobi (including the special cases named after Chebyshev, Legendre and Gegenbauer) are the most extensively studied and widely applied systems. These four families of orthogonal polynomials are called collectively the '*classical orthogonal polynomials*'.

In 1929, Bochner posed a problem of determining all families of scalar-valued orthogonal polynomials that are eigenfunctions of some fixed second-order linear differential operator. This problem was solved by Bochner in the original paper [1], and was considered many times later, for example by Grünbaum and Haine in [2]. The only families of orthogonal polynomials that are eigenfunctions of some fixed second-order linear differential operator

$$\mathcal{F}P_n(x) = \lambda_n P_n(x)$$

are the classical ones with the classical differential operators $\mathcal{F}$ listed below:

(i) *Jacobi*. The classical differential operator is

$$\mathcal{F} = x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\alpha + 1 - (\alpha+\beta+2)x)\frac{\mathrm{d}}{\mathrm{d}x}, \tag{1}$$

with

$$\lambda_n = -n(n+\alpha+\beta+1) \qquad \text{and} \qquad \alpha, \beta > -1;$$

(ii) *Bessel*. The classical differential operator is

$$\mathcal{F} = x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2x-1)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \text{with} \quad \lambda_n = n(n+1); \tag{2}$$

(iii) *Hermite*. The classical differential operator is

$$\mathcal{F} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \text{with} \quad \lambda_n = -2n; \tag{3}$$

(iv) *Laguerre*. The classical differential operator is

$$\mathcal{F} = x \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\alpha+1-x)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \text{with} \quad \lambda_n = -n. \tag{4}$$

A natural question is to classify all differential operators which have Jacobi, Bessel, Hermite or Laguerre polynomials as eigenfunctions. This issue is addressed in the next section.

2. Classical orthogonal polynomials and differential operators

In this section linear differential operators having classical orthogonal polynomials as eigenfunctions are described.

The following lemma summarizes some preliminary facts that will be used later in the paper.

Lemma 1. *Given a family of orthogonal polynomials* $\{P_n(x)\}_{n=0}^{\infty}$ *and a linear differential operator* $\mathcal{U}$ *of order* s *such that*

$$\mathcal{U}P_n(x) = \Gamma_n P_n(x), \tag{5}$$

where

$$\mathcal{U} = \sum_{i=0}^{s} f_i(x)\frac{\mathrm{d}^i}{\mathrm{d}x^i}; \tag{6}$$

$f_i(x)$ *are polynomials in* x *of degree* i *and* Γ_n *depend on* n *but not on* x. *Then*

(i) the operator $\mathcal{U}$ *has to be of even order, i.e.* $s = 2k$;
(ii) the operator $\mathcal{U}$ *is uniquely defined by* $P_n(x)$ *and* Γ_n *for* $n = 0, \ldots, s$.

Proof.

(i) It was proved by Krall in [4] that there is no differential operator of type (6) of odd order which has orthogonal polynomials as solutions, hence $s = 2k$.
(ii) Starting from $j = 0$ one can go up to $j = s$ and substitute $P_j(x)$ into (5). Comparing coefficients going with the powers x^m for all $m = 0, \ldots, j$ determines $f_j(x)$. □

Theorem 1. *Given a family of classical (Jacobi, Bessel, Hermite or Laguerre) orthogonal polynomials* $\{P_n(x)\}_{n=0}^{\infty}$. *Suppose there exists a differential operator*

$$\mathcal{U} = \sum_{i=0}^{2k} f_i(x)\frac{\mathrm{d}^i}{\mathrm{d}x^i}, \qquad k > 1,$$

such that

$$\mathcal{U}P_n(x) = \Gamma_n P_n(x), \tag{7}$$

where Γ_n is a function of n but not x. Then

$$\mathcal{U} = \sum_{j=0}^{k} c_j \mathcal{F}^j,$$

where $c_j \in \boldsymbol{C}$ and the second-order differential operator $\mathcal{F}$ such that

$$\mathcal{F} P_n(x) = \lambda_n P_n(x), \tag{8}$$

is the classical differential operator associated with each family, see (1)–(4).

Proof. First we will show that the coefficients $f_j(x)$ are polynomials of degree $\leqslant j$. Following the discussion of Bochner in [1], we substitute $P_0(x)$ into (7) and conclude that $f_0(x)$ has to be a constant function in x. Substituting $P_1(x)$ into (7) shows that $f_1(x)$ has to be linear in x. By continuing this process one concludes that $f_i(x)$ has to be a polynomial of degree at most i for all $i = 0, \ldots, 2k$. Denote $f_i(x) = \sum_{j=0}^{i} a_j^i x^j$. After substituting a polynomial $P_n(x)$ into (7) and comparing coefficients going with the x^n one obtains that

$$\Gamma_n = (n) \cdots (n - 2k + 1)\, a_{2k}^{2k} + (n) \cdots (n - 2k + 2)\, a_{2k-1}^{2k-1} + \cdots + a_0^0.$$

To simplify the notation, denote

$$(m)_n := m(m+1) \cdots (m - n + 1); \qquad (m)_0 = 1. \tag{9}$$

Case 1. λ_n is linear in n (Hermite, Laguerre). Suppose Γ_n is of some degree s such that $s > k$ (or $s < k$). Since λ_n is linear in n, there must exist coefficients c_j such that $\Gamma_n = \sum_{j=0}^{s} c_j \lambda_n^j$, where $c_s \neq 0$. The differential operator $\mathcal{R} = \sum_{j=0}^{s} c_j \mathcal{F}^j$ has order exactly $2s > 2k$ (or $2s < 2k$) and it is easy to see that $\mathcal{R} P_n(x) = \Gamma_n P_n(x)$. However, in lemma 1 it was observed that the differential operator is uniquely defined by the polynomials $P_n(x)$ and Γ_n, which implies that $\mathcal{R}$ must be identical to $\mathcal{U}$ and $s = k$. Since $\mathcal{R} \equiv \mathcal{U}$, then Γ_n must be of degree exactly k, implying that

$$\Gamma_n = \sum_{j=0}^{k} c_j \lambda_n^j \qquad \text{and} \qquad \mathcal{U} = \sum_{j=0}^{k} c_j \mathcal{F}^j.$$

Case 2. λ_n is quadratic in n (Jacobi, Bessel). Let us introduce a new differential operator $\mathcal{U}_1 = \mathcal{U} - c_k \mathcal{F}^k$. The coefficient c_k is chosen in such a way that the coefficient in front of the term $\mathrm{d}^{2k}/\mathrm{d}x^{2k}$ of the operator $\mathcal{U}_1$ is a polynomial of degree at most $2k-1$ instead of $2k$. It is possible to do so since the operator $\mathcal{F}^s$ is an operator of order $2s$ with the polynomial of degree exactly $2s$ in front of the term $\mathrm{d}^{2s}/\mathrm{d}x^{2s}$ for any s. The same procedure can be repeated to obtain an operator $\mathcal{U}_2 = \mathcal{U}_1 - c_{k-1}\mathcal{F}^{k-1}$, with the coefficient in front of the term $\mathrm{d}^{2k-2}/\mathrm{d}x^{2k-2}$ of the operator $\mathcal{U}_2$ being a polynomial of degree at most $2k-3$. Continuing this process one arrives at

$$\mathcal{U}_k = \mathcal{U} - \sum_{j=0}^{k} c_j \mathcal{F}^j \qquad \text{and} \qquad \bar{\Gamma}_n = \Gamma_n - \sum_{j=0}^{k} c_j \lambda_n^j$$

such that

$$\mathcal{U}_k P_n(x) = \bar{\Gamma}_n P_n(x). \tag{10}$$

Denote the coefficient associated with the term $\mathrm{d}^j/\mathrm{d}x^j$ $(j = 0, \ldots, 2k)$ of the operator $\mathcal{U}_k$ to be a polynomial $f_j(x) = \sum_{i=0}^{j} a_i^j x^i$ of degree at most j. By our construction above, the coefficient in front of the term $\mathrm{d}^{2j}/\mathrm{d}x^{2j}$ is a polynomial of degree $2j-1$, i.e. $a_{2j}^{2j} = 0$ for all $j = 0, \ldots, k$. Note that as a result of the procedure described above, $\bar{\Gamma}_n$ as a polynomial in n

has only odd powers of n. Below it will be shown that $\bar{\Gamma}_n \equiv 0$, which will imply that $\mathcal{U}_k \equiv 0$, hence $\mathcal{U} = \sum_{j=0}^{k} c_j \mathcal{F}^j$.

Denote the nth Jacobi (or Bessel) polynomial

$$P_n(x) = \sum_{j=0}^{n} B_j x^j. \tag{11}$$

Substitute expression (11) into (10) and collect the terms going with each x^m for $m = 0, \ldots, n$ to obtain a $(2k+1)$-term recursion relation for the coefficients B_j, where notation (9) is used:

$$\begin{aligned}(m+2k)_{2k} a_0^{2k} B_{m+2k} &+ (m+2k-1)_{2k-1}\left(m a_1^{2k} + a_0^{2k-1}\right) B_{m+2k-1} + \cdots \\ &+ (m+1)\left((m)_{2k-1} a_{2k-1}^{2k} + (m)_{2k-2} a_{2k-2}^{2k-1} + \cdots + a_0^1\right) B_{m+1} \\ &+ \left((m)_{2k} a_{2k}^{2k} + (m)_{2k-1} a_{2k-1}^{2k-1} + \cdots + a_0^0 - \bar{\Gamma}_n\right) B_m = 0.\end{aligned} \tag{12}$$

By substituting expression (11) into (8) where the classical differential operator $\mathcal{F}$ corresponds to the family of the Jacobi polynomials (see [1]), one obtains another recursion relation for the coefficients B_j:

$$B_{m+1} = B_m \frac{m(m+\alpha+\beta+1) - n(n+\alpha+\beta+1)}{(m+1)(m+\alpha+1)}. \tag{13}$$

From the expression above it follows that for any $s \geqslant 1$

$$B_{m+s} = B_m \frac{v_m \cdots v_{m+s-1}}{(m+s)_s (m+\alpha+s)_s}, \tag{14}$$

where $v_m = m(m+\alpha+\beta+1) - n(n+\alpha+\beta+1)$. After substituting (14) into (12), one obtains

$$\begin{aligned}\frac{v_m \cdots v_{m+2k-1}}{(m+\alpha+2k)_{2k}} a_0^{2k} B_m &+ \left(m a_1^{2k} + a_0^{2k-1}\right) \frac{v_m \cdots v_{m+2k-2}}{(m+\alpha+2k-1)_{2k-1}} B_m + \cdots \\ &+ \left((m)_{2k-1} a_{2k-1}^{2k} + (m)_{2k-2} a_{2k-2}^{2k-1} + \cdots + a_0^1\right) \frac{v_m}{(m+\alpha+1)} B_m \\ &+ (\bar{\Gamma}_m - \bar{\Gamma}_n) B_m = 0.\end{aligned}$$

After dividing by B_m and reducing the expression above to the common denominator one arrives at:

$$\begin{aligned}v_m \cdots v_{m+2k-1} a_0^{2k} &+ v_m \cdots v_{m+2k-2} (m+\alpha+2k)_1 \left((m)_1 a_1^{2k} + a_0^{2k-1}\right) + \cdots \\ &+ v_m (m+\alpha+2k)_{2k-1} \left((m)_{2k-1} a_{2k-1}^{2k} + (m)_{2k-2} a_{2k-2}^{2k-1} + \cdots + a_0^1\right) \\ &+ (m+\alpha+2k)_{2k} (\bar{\Gamma}_m - \bar{\Gamma}_n) = 0.\end{aligned} \tag{15}$$

Note that $v_m \cdots v_{m+2k-s}$ is a polynomial in n of degree $2(2k-s+1)$; hence expression (15) is a polynomial in n of degree $4k$ where the n dependence comes from the terms v_j and $\bar{\Gamma}_n$, where $\bar{\Gamma}_n$ is at most of degree $2k$. This implies that the coefficients in front of $v_m \cdots v_{m+s}$ for $s = k, \ldots, 2k-1$ are zero. Now the highest degree of n in expression (15) is $2k$ and the coefficient in front of n^{2k} is

$$(m+\alpha+2k)_k \left((m)_k a_k^{2k} + \cdots + a_0^k\right) - (m+\alpha+2k)_{2k} a_{2k}^{2k} = 0.$$

By our construction $a_{2k}^{2k} = 0$; hence the coefficient in front of $v_m \cdots v_{m+k-1}$ is zero. The next highest degree of n in expression (15) is $2k-1$ and it comes from $\bar{\Gamma}_n$ since $v_m \cdots v_{m+k}$ is of degree $2k-2$. This implies that a_{2k-1}^{2k-1} must be zero, which means that the polynomial $\bar{\Gamma}_n$ must be of degree at most $2k-2$. By repeating the argument above and using the fact that $a_{2k-2}^{2k-2} = 0$, we conclude that $\bar{\Gamma}_n$ must be of degree at most $2k-4$ and by continuing this process we arrive at the conclusion that $\bar{\Gamma}_n \equiv 0$.

In case of the Bessel polynomials the recursion relation (13) becomes (see [1])

$$B_{m+1} = B_m \frac{m(m+1) - n(n+1)}{m+1}.$$

Expression (15) is now

$$v_m \cdots v_{m+2k-1} a_0^{2k} + v_m \cdots v_{m+2k-2}\big(m a_1^{2k} + a_0^{2k-1}\big) + \cdots \\ + v_m\big((m)_{2k-1} a_{2k-1}^{2k} + (m)_{2k-2} a_{2k-2}^{2k-1} + \cdots + a_0^1\big) + (\bar{\Gamma}_m - \bar{\Gamma}_n) = 0.$$

An argument identical to the one presented for the Jacobi case applies to the Bessel case allowing one to conclude that $\bar{\Gamma}_n \equiv 0$. This implies that $\mathcal{U}_k \equiv 0$; hence

$$\mathcal{U} = \sum_{j=0}^{k} c_j \mathcal{F}^j,$$

which concludes the proof of the theorem. □

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